COMMON SOLUTIONS OF TWO SIMULTANEOUS PELL EQUATIONS.

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We shall in this brief paper discuss the two Pell equations

$$x^2 - 2y^2 = 1, \qquad y^2 - 3z^2 = 1 \tag{1}$$

relative to their common integral solutions. That $x = 3$, $y = 2$, $z = 1$ is such a solution we see immediately, and ask then: Do other integral solutions exist?

To answer this question we subtract one of our equations from the other, and get

$$x^2 - 3y^2 + 3z^2 = 0. \tag{2}$$

Every solution of this equation* may according to the general theory of numbers of the domain $K(\sqrt{-3})$ be written in the form

$$x = 3pq, \qquad y = \tfrac{1}{2}(3p^2 + q^2), \qquad z = \pm\,\tfrac{1}{2}(3p^2 - q^2), \tag{3}$$

where the double sign of z will be explained immediately. Introducing these values of y and z in (1) we get

$$q^4 - 12p^2q^2 + 9p^4 = -2. \tag{4}$$

The solutions of the second equation (1) are given by the equation

$$(y + z\sqrt{3}) = (2 + \sqrt{3})^r. \tag{5}$$

If $r \equiv 0 \pmod 3$ were possible, then $z \equiv 0 \pmod 3$, and hence from (3) $q \equiv 0$. This, however, contradicts equation (4).

When $r = 3s_1 + 1$ we have

$$\begin{aligned} y + z\sqrt{3} &= (2 + \sqrt{3})^{3s_1+1}, \\ (2 - \sqrt{3})(y + z\sqrt{3}) &= (2 + \sqrt{3})^{3s_1}, \\ (2y - 3z) + \sqrt{3}(2z - y) &= (2 + \sqrt{3})^{3s_1}, \end{aligned} \tag{6}$$

or

$$2z - y \equiv -(z + y) \equiv 0 \pmod 3$$

from which follows that the sign $+$ must be used in the value for z in equation (3). When $r = 3s_2 - 1$ it follows in the same way that the sign $-$ must be used. Both of these cases satisfy equation (4).

* See for example Bachmann, P., Niedere Zahlentheorie, vol. II, p. 456.

Upon a closer examination of (4) we find in the first place that in the number domain $K(\sqrt{3})$ it may be factored as follows:

$$(q^2 - 6p^2 + 3\sqrt{3}\,p^2)(q^2 - 6p^2 - 3\sqrt{3}\,p^2) = (-2). \tag{7}$$

From

$$(q^2 - 6p^2 + 3\sqrt{3}\,p^2) = (q - \sqrt{3}\sqrt{2 - \sqrt{3}}\,p)(q + \sqrt{3}\sqrt{2 - \sqrt{3}}\,p) \tag{8}$$

follows then its final division into factors in the number domain $K(\sqrt{2 - \sqrt{3}})$, which is a relative domain of $K(\sqrt{3})$ constructed on the unity $2 - \sqrt{3}$. This is a Galois domain which, on account of the relations

$$\sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}} = \sqrt{2}, \qquad \sqrt{2 + \sqrt{3}} + \sqrt{2 - \sqrt{3}} = \sqrt{6}, \tag{9}$$

is identical with the domain $K(\sqrt{2}, \sqrt{3})$ constructed from $K(\sqrt{2})$ and $K(\sqrt{3})$. Its defining equation may be written in the form

$$x^4 - 4x^2 + 1 = 0, \tag{10}$$

and a base is given in $1, \sqrt{3}, \sqrt{2 - \sqrt{3}}, \sqrt{3}\sqrt{2 - \sqrt{3}}$, which leads to the discriminant $d = 2^8 \cdot 3^2$ of the domain. To decide on the number of ideal classes in $K(\sqrt{2 - \sqrt{3}})$ it is only necessary to examine the ideals whose norm* is $\leq \frac{4!}{4^4}\sqrt{d} = \frac{9}{2}$, i.e., the two prime numbers 2 and 3.

In $K(\sqrt{3})$ we have

$$(2) = (\sqrt{3} + 1)(\sqrt{3} - 1), \qquad (3) = (\sqrt{3})(\sqrt{3}), \tag{11}$$

where the parentheses indicate that there is a question of division into ideals. In $K(\sqrt{2 - \sqrt{3}})$ we have

$$(\sqrt{3} - 1) = (1 - \sqrt{2 - \sqrt{3}})(1 + \sqrt{2 - \sqrt{3}})$$

from which we conclude that only one ideal class exists, which is the principal ideal class. From the ideal equation

$$\begin{aligned}(-2) = (1 - \sqrt{3}\sqrt{2 - \sqrt{3}})&(1 + \sqrt{3}\sqrt{2 - \sqrt{3}}) \\ &\times (\sqrt{2 - \sqrt{3}} - \sqrt{3})(\sqrt{2 - \sqrt{3}} + \sqrt{3})\end{aligned}$$

follows on account of (7) and (8) a number identity of one of the two types:

$$\begin{gathered}[1 - \sqrt{3}\sqrt{2 - \sqrt{3}}][m_1 + m_2\sqrt{3} + (n_1 + n_2\sqrt{3})\sqrt{2 - \sqrt{3}}] \\ \equiv [q - p\sqrt{3}\sqrt{2 - \sqrt{3}}] \\ \text{or} \\ [1 - \sqrt{3}\sqrt{2 - \sqrt{3}}][m_1 + m_2\sqrt{3} + (n_1 + n_2\sqrt{3})\sqrt{2 - \sqrt{3}}] \\ \equiv [q\sqrt{2 - \sqrt{3}} - p\sqrt{3}],\end{gathered} \tag{12}$$

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* Minkowski, H., Diophantische Approximationen, 1907, Theorem LIX, page 185.

where the expression in the second parenthesis represents a unit in $K(\sqrt{2 - \sqrt{3}})$, and m_1, m_2, n_1, n_2, p and q are rational integers. From the general theory of number domains* we know of the existence in $K(\sqrt{2 - \sqrt{3}})$ of three so-called fundamental units ϵ_1, ϵ_2, and ϵ_3, which have the property that every other unit E in $K(\sqrt{2 - \sqrt{3}})$ may be written in the form:

$$E = \pm \epsilon_1^m \epsilon_2^n \epsilon_3^r, \tag{13}$$

m, n, and r being integers. In the case of $K(\sqrt{2 - \sqrt{3}})$ we may for example take $\epsilon_1 = \sqrt{2 + \sqrt{3}}$, $\epsilon_2 = 1 + \sqrt{2}$, $\epsilon_3 = \sqrt{3} + \sqrt{2}$, and have then

$$\begin{aligned} E = \pm (\sqrt{2 + \sqrt{3}})^{\eta_1}(1 + \sqrt{2})^{\eta_2}(\sqrt{3} + \sqrt{2})^{\eta_3}(A + B\sqrt{3}) \\ \times (M + N\sqrt{2})(P + Q\sqrt{6}), \end{aligned} \tag{13'}$$

where the exponents η_1, η_2, η_3 independently of each other may take the values 0 or 1, and for which the equations

$$A^2 - 3B^2 = 1, \qquad M^2 - 2N^2 = 1, \qquad P^2 - 6Q^2 = 1, \tag{14}$$

are satisfied. We have the following relations:

$$\begin{aligned} &[\sqrt{2 + \sqrt{3}}]^2 = 2 + \sqrt{3} = [2 + \sqrt{2 - \sqrt{3}}][2 - \sqrt{2 - \sqrt{3}}], \\ &[\sqrt{3} + \sqrt{2}]^2 = 5 + 2\sqrt{6} = [2 + \sqrt{2 + \sqrt{3}}][2 + \sqrt{2 - \sqrt{3}}], \\ &[1 + \sqrt{2}]^2 = 3 + 2\sqrt{2} = [2 + \sqrt{2 + \sqrt{3}}][2 - \sqrt{2 - \sqrt{3}}], \\ &[1 + \sqrt{2}][\sqrt{3} + \sqrt{2}] \\ &\qquad = 2 + \sqrt{3} + [\sqrt{6} + \sqrt{2}] = \sqrt{2 + \sqrt{3}}[2 + \sqrt{2 + \sqrt{3}}], \end{aligned} \tag{15}$$

which are not without importance.

From (12) the following system of equations is obtained:

$$\begin{array}{rcccc} 1\cdot m_1 + 0\cdot m_2 + 3n_1 - 6n_2 = & q(1) & \text{or} & 0(2) \\ 0\cdot m_1 + 1\cdot m_2 - 2n_1 + 3n_2 = & 0 & `` & -p \\ 0\cdot m_1 - 3m_2 + 1\cdot n_1 + 0\cdot n_2 = & 0 & `` & q \\ -1\cdot m_1 + 0\cdot m_2 + 0\cdot n_1 + 1\cdot n_2 = & -p & `` & 0 \end{array} \tag{16}$$

that is to say in case (1)

$$m_1 = \frac{5q - 3p}{2}, \qquad m_2 = 3\cdot\frac{q - p}{2}, \qquad n_1 = 9\cdot\frac{q - p}{2}, \qquad n_2 = 5\frac{q - p}{2},$$

or the two relations:

$$n_1 = 3m_2, \qquad 3n_2 = 5m_2, \tag{17}$$

* Hilbert, D., "Die Theorie der algebraischen Zahlkörper," Ber. der Deutsch. Math.-Verein., 1897, p. 214, Theorem 47.

and in case (2)

$$m_1 = 3\frac{q-3p}{2}, \qquad m_2 = \frac{q-5p}{2}, \qquad n_1 = 5\cdot\frac{q-3p}{2}, \qquad n_2 = 3\cdot\frac{q-3p}{2},$$

or

$$n_2 = m_1, \qquad 3n_1 = 5m_1. \tag{18}$$

Furthermore we have

$$[m_1^2 + 3m_2^2 - 2n_1^2 - 6n_2^2 + 6n_1n_2]^2 - 3[2m_1m_2 - 4n_1n_2 + n_1^2 + 3n_2^2]^2 = 1.$$

By comparing the coefficients m_1, m_2, n_1, and n_2 in

$$m_1 + m_2\sqrt{3} + [n_1 + n_2\sqrt{3}]\sqrt{2-\sqrt{3}} \tag{19}$$

with those in (13′) we obtain the former expressed as functions of A, B, M, N, P, and Q. The different combinations $\eta_1, \eta_2, \eta_3 = 0$ or 1 must in this connection be treated separately. In the systems of equations (14) and (17) or (14) and (18) we have thus five equations with six unknown quantities. The purpose of this paper is now to show how a sixth independent relation may be found, by means of which an algebraic equation in one of the quantities A, B, etc., is obtained, and which equation we then shall have to examine only with reference to possible integral solutions. We shall in the following deal only with the case $\eta_1 = \eta_2 = \eta_3 = 0$. When we perform the substitution $(\sqrt{3};\ -\sqrt{3})$ on E we find that on account of (9) $\sqrt{6}$ remains unchanged, while $\sqrt{2}$ changes into $\sqrt{-2}$, and a unit E_1 results which has the form:

$$E_1 = \pm(\sqrt{2-\sqrt{3}})^{\eta_1}(1-\sqrt{2})^{\eta_2}(-1)^{\eta_3}(\sqrt{3}+\sqrt{2})^{\eta_3} \times (A - B\sqrt{3})(M - N\sqrt{2})(P + Q\sqrt{6}),$$

i.e.,

$$E\cdot E_1 = (-1)^{\eta_2+\eta_3}(5+2\sqrt{6})^{\eta_3}(P+Q\sqrt{6})^2. \tag{20}$$

If the same substitution is performed on (19), and if the values $m_1 = \alpha$, $m_2 = 3\beta$, $n_1 = 9\beta$, $n_2 = 5\beta$ are introduced, we get

$$E\cdot E_1 = (\alpha^2 - 21\beta^2) + \sqrt{6}(4\alpha\beta - 18\beta^2);$$

that is to say, when only the case $\eta_1 = \eta_2 = \eta_3 = 0$

$$P^2 + 6Q^2 = \alpha^2 - 21\beta^2, \qquad P^2 - 6Q^2 = 1$$

is considered, we get

$$2P^2 = \alpha^2 - 21\beta^2 + 1$$

or

$$6P^2 = 3m_1^2 - 7m_2^2 + 3. \tag{21'}$$

If the substitution $(\sqrt{2-\sqrt{3}};\ -\sqrt{2-\sqrt{3}})$ is used, $\sqrt{6}$ changes into $-\sqrt{6}$, $\sqrt{2}$ into $-\sqrt{2}$, while $\sqrt{3}$ remains unchanged. Hence,

$$E\cdot E_2 = (-1)^{\eta_1+\eta_2}(2+\sqrt{3})^{\eta_1}(A+B\sqrt{3})^2,$$
$$E\cdot E_2 = (\alpha^2-15\beta^2)+\sqrt{3}(6\alpha\beta-24\beta^2),$$

or

$$A^2+3B^2=\alpha^2-15\beta^2,\qquad A^2-3B^2=1,$$
$$2A^2=\alpha^2-15\beta^2+1,\qquad 6A^2=3m_1{}^2-5m_2{}^2+3. \tag{21''}$$

The two substitutions $(\sqrt{3};\ -\sqrt{3})$ and $(\sqrt{2-\sqrt{3}};\ -\sqrt{2-\sqrt{3}})$ used simultaneously give us

$$E\cdot E_3 = (-1)^{\eta_1+\eta_3}(3+2\sqrt{2})^{\eta_2}(M+N\sqrt{2})^2,$$
$$E\cdot E_3 = (\alpha^2-33\beta^2)+\sqrt{2}(6\alpha\beta-36\beta^2),$$

or

$$6M = 3m_1{}^2-11m_2{}^2+3. \tag{21'''}$$

Eliminating m_1 and m_2 from the three equations (21′), (21″), and (21‴) we get

$$3P^2-2A^2=M_1{}^2 \tag{22}$$

which for case (1), in which $\eta_1=\eta_2=\eta_3=0$, gives us the sixth independent equation. To show that this really is the case, we eliminate the four variables B, M, N, and Q, and obtain the two equations:

$$\begin{aligned}-4A^8-48A^6P^2+32A^6+132A^4P^4-60A^4P^2-64A^4-72A^2P^6\\ -24A^2P^4+144A^2P^2-9P^8+54P^6-81P^4=0,\end{aligned} \tag{23'}$$

and

$$\begin{aligned}-4A^8P^8-64A^8P^6+80A^8P^4+96A^8P^2-144A^8+8A^6P^8\\ +272A^6P^6-376A^6P^4+96A^6P^2-144A^6+32A^4P^8-28A^4P^6\\ +296A^4P^4-192A^4P^2-36A^4-36A^2P^8-828A^2P^6+972A^2P^4\\ +324A^2P^2-81P^8+486P^6-729P^4=0.\end{aligned} \tag{24'}$$

From (23′) we see then in the first place that no factor can be found which is independent of A. Furthermore, if (23′) were reducible, it must remain so for any arbitrary value of P^2, for example for $P^2=-1$. For this value of P^2 (23′) may be reduced to the form

$$A^8-20A^6-32A^4+24A^2+36=0, \tag{23''}$$

which by a simple discussion may be shown to be irreducible. For the same value $P^2=-1$ of P we obtain from (24′) the equation

$$25A^8+220A^6-128A^4-360A^2+324=0 \tag{24''}$$

from which it is seen that (23′) and (24′) really are distinct equations, and that the elimination from these of, for example, P^2 will lead to the desired

algebraic equation in A^2. This equation must then be discussed for possible integral solutions, which in the first place must satisfy equations (14). Finally we may easily verify that (23′) and (24′) are indeed satisfied by $A^2 = P^2 = 1$, which give us the already known solution $p = q = 1$.

In this way every combination $\eta_1, \eta_2, \eta_3 = 0$ or 1 must be tried in the two cases (1) and (2). Thus we find that our problem is completely solved by a finite number of purely algebraic operations. It is possible that a discussion of (22), (14), and (17) with reference only to divisibility would show that no other solution than the one mentioned could exist, and that thus in this special case the long process of elimination could be obviated. A similar method may be applied on equations of the type

$$ax^4 + 2bx^2y^2 + cy^4 = A,$$

where a, b, c, and A are given integers, whenever the ultimate relative domain is a Galois domain, as in the above example.

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